Modal Coupling in Thermally Stressed Plates

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An approximate, but general, solution for the frequencies, thus the effective stiffnesses, in the first and second modes of initially deformed, thermally stressed plates of any planform shape and with any boundary condition is found in terms of quantities that may be obtained from the application of linear theory. It is shown that all plates exhibit the same characteristic changes in frequency, thus stiffness, independent of planform shape, boundary conditions and temperature distribution except as these factors affect the thermal buckling eigenvalues (ΔT critical) and the natural, uniform temperature frequencies. Coupling between the modes is shown to depend upon the ratio of the thermal buckling eigenvalues and the uniform temperature frequency ratio. Analytical data are presented to show that the second mode stiffness for a plate for certain values of the parameters does not always increase in the postbuckled region as implied in the literature. Experimental data, obtained from cantilever plates of different planforms, are presented to show that mode coupling is readily detected and cannot be neglected if a correct prediction of the effective stiffness of a plate is desired. The analysis may readily be extended to any desired number of modes. Only a qualitative comparison is made between theory and experiment in this paper. A carefully controlled test program will be required to produce data for a quantitative comparison.

Nomenclature

B = normal coordinate for the first mode
C = large deflection stress function parameter
D = plate stiffness, $Et^3/12(1 - \nu^2)$
E = modulus of elasticity
F = large deflection stress function
G = small deflection stress function
$I_1, \ldots, I_9$ = definite integrals involving functions W_1, W_2, and G
I_{10} = work done by the stress over that part of the boundary on which the displacements are prescribed
T = temperature distribution over the surface of the plate
ΔT_1 = reference temperature at which buckling in first mode occurs
ΔT_2 = reference temperature at which buckling in second mode occurs
t = plate thickness
$\bar{U}, \bar{V}$ = displacement in the x and y directions, respectively, on that part of the boundary where the displacements are prescribed
W = total displacement of elastic surface from the x,y plane
$\ddot{W}$ = second derivative with respect to time
W_i = initial displacement of elastic surface from the x, y plane; also called initial imperfection and/or initial deformation
W_1 = first mode from linear solution
W_2 = second mode from linear solution
Z = sum of forces per unit area normal to the plane of the plate, $P(x,y,\tau) - \rho t\ddot{W}$
$P(x,y,\tau)$ = applied load over plate surface
α = thermal coefficient of expansion
Γ = energy due to heating or thermal loading $\iint t\alpha T(\sigma_x + \sigma_y)dxdy$
Γ_1 = value of Γ at which the perfect, unloaded plate would buckle in the first mode
Γ_2 = value of Γ at which the perfect, unloaded plate would buckle in the second mode
ϵ_ψ = small amplitude dynamic displacement of first mode
ϵ_ϕ = small amplitude dynamic displacement of second mode
θ = normal coordinate for second mode
ν = Poisson's ratio
ρ = density per unit volume
τ = time
ϕ = nondimensionalized normal coordinate for second mode
ψ = nondimensionalized normal coordinate for first mode
Φ = nondimensional large static deflection, second mode
Ψ = nondimensional large static deflection, first mode
ω = frequency of small amplitude vibration about large amplitude static equilibrium position
ω_{01} = first mode, free vibration frequency, at uniform temperature
ω_{02} = second mode, free vibration frequency, at uniform temperature

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Introduction

HELDENFELS and Vosteen[2] showed that the second mode (torsional) frequency and stiffness of a thermally stressed square cantilever plate always increases after reaching some minimum value as the temperature increases. The minimum stiffness is dependent upon the initial deformation. The solution obtained is shown in Fig. 1, where the parameter ϕ_i is a measure of the initial twist in the plate. They tested a square cantilever plate that had initial deformation in both the first and second modes. The plate and its initial shape are shown in Fig. 2. Their plot of experimental data which verified their analytical solution is shown in Fig. 3. Breuer[4] verified the results of Heldenfels and Vosteen and extended the solution to plates of other aspect ratio.

Bailey[3] extended the analytical results of Heldenfels and Vosteen and showed that their solution for the second mode is also the solution for the first mode when the two modes are uncoupled. It can readily be shown by use of the theory of orthonormal functions that it is the solution for any uncoupled mode. However, in conducting an experimental investigation of tapered plates in the prebuckled region, it was noted that in those instances when the heat was left on

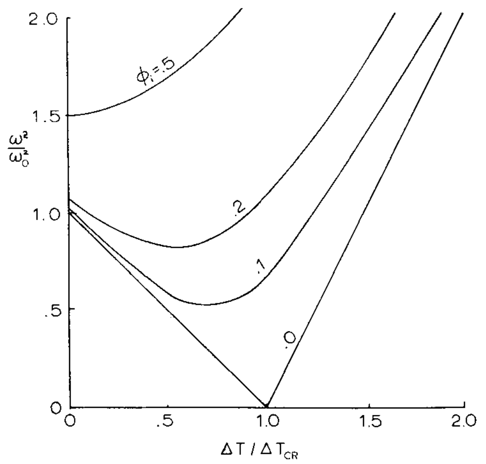

Fig. 1 Response of uncoupled modes.[2]

longer than usual, the second mode frequency did not increase after reaching a minimum but leveled off, indicating that the stiffening effect as recorded by others was not present. The fully tapered plate was the only exception. No attempt was made to explain the cause of this phenomena until more recent work was accomplished. That work is reported herein.

The terminology, first and second modes, is used to imply the generality of the solution to both symmetrical and unsymmetrical plates. "Bending" and "torsion" could be used, but these terms apply, in a strict sense, to symmetrical plates where in linear theory the symmetrical (bending) and antisymmetrical (torsion) deflections are completely uncoupled. For an unsymmetrical plate in linear theory and for all plates in nonlinear theory, all modes are plate bending modes containing both symmetrical and antisymmetrical deflections.

Statement of the Problem and Basic Equations

Given an initially deformed plate of any planform shape, boundary conditions, and thickness distribution (provided the thickness distribution permits the assumption of thin plate theory), find the frequency, thus the effective stiffness, when the plate is subjected to thermal stress and large deflections. It is further assumed that the linear solutions to

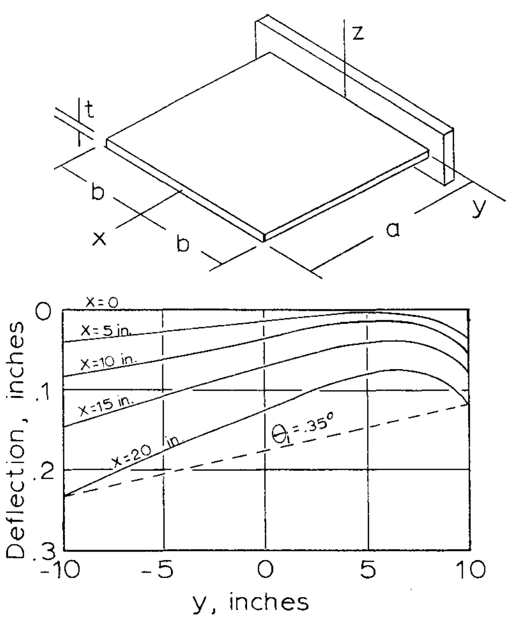

Fig. 2 Initial plate shape.[2]

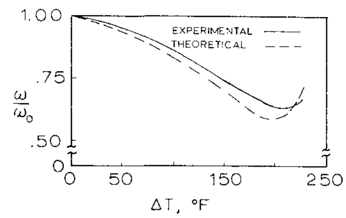

Fig. 3 Frequency response.[2]

the small deflection problem of the perfect plate for specified boundary conditions, thickness distribution and planform shape are known. (By linear solutions is meant the small deflection solution to the vibrations problem when no thermal stresses are present, and the solution to the in-plane thermal stress problem when no displacements normal to the plane are present.)

Since there is no exact solution to the differential equations of this problem, an approximate solution is obtained by using Reissner's Variational Principle[1] for large deflections:

$$\delta\Bigg[\iint\Bigg[\frac{D}{2}\left\{\left[\frac{\partial^2(W-W_i)}{\partial x^2}\right]^2+\left[\frac{\partial^2(W-W_i)}{\partial y^2}\right]^2+2\nu\,\frac{\partial^2(W-W_i)}{\partial x^2}\,\frac{\partial^2(W-W_i)}{\partial y^2}+2(1-\nu)\times\left[\frac{\partial^2(W-W_i)}{\partial x\partial y}\right]^2\right\}+\frac{t}{2}\left\{\frac{\partial^2F}{\partial x^2}\left[\left(\frac{\partial W}{\partial y}\right)^2-\left(\frac{\partial W_i}{\partial y}\right)^2\right]+\frac{\partial^2F}{\partial y^2}\left[\left(\frac{\partial W}{\partial x}\right)^2-\left(\frac{\partial W_i}{\partial x}\right)^2\right]-2\,\frac{\partial^2F}{\partial x\partial y}\left[\frac{\partial W}{\partial x}\,\frac{\partial W}{\partial y}-\frac{\partial W_i}{\partial x}\,\frac{\partial W_i}{\partial y}\right]\right\}-\frac{t}{2E}\left\{\left(\frac{\partial^2F}{\partial x^2}\right)^2+\left(\frac{\partial^2F}{\partial y^2}\right)^2-2\nu\,\frac{\partial^2F}{\partial x^2}\,\frac{\partial^2F}{\partial y^2}+2(1+\nu)\left(\frac{\partial^2F}{\partial x\partial y}\right)^2\right\}-t\alpha\,T(x,y,\tau)\times\left\{\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}\right\}\Bigg]dxdy-\iint[P(x,y,\tau)-\rho t\ddot{W}]\times W dxdy+\int_c\left[\frac{\partial^2F}{\partial y^2}\,\bar{U}+\frac{\partial^2F}{\partial x^2}\,\bar{V}\right]ds\Bigg]=0$$

In the application of this minimal energy principle the generality of the large deflection solution obtained is a result of the orthonormal functions chosen for the deflection function. Two functions must be assumed; the deflection function W and the stress function F.

The deflection function is assumed to be

$$W(x,y,\tau) = B(\tau)W_1(x,y) + \theta(\tau)W_2(x,y)$$

where $W_1(x,y)$ and $W_2(x,y)$ are the first and second buckling or vibration modes, as determined from the linear solution of whatever plate problem happens to be prescribed. Thus, these functions are orthogonal and satisfy the plate displacement boundary conditions. B and θ are undetermined parameters, in this case normal coordinates, for the modes W_1 and W_2. Although it is arbitrary, W_i, the initial displacement, is taken to be the same functional form as the large deflection mode

$$W_i(x,y) = B_iW_1(x,y) + \theta_iW_2(x,y)$$

B_i and θ_i are the measured amplitudes that define the magnitude of the initial displacement.

The stress function for large deflections is assumed to be

$$F = C(\tau)G(x,y)$$

where $G(x,y)$ is the stress function obtained from a linear solution of the stress distribution; thus, it satisfies the stress boundary conditions for whatever planform shape the plate may have. C, B, and θ are undetermined parameters that must be found from the application of Reissner's variational principle.

Substitution of the assumed functions into the variational equation yields three equations in the three unknowns:

$$\ddot{B} + (I_2/I_1)(B - B_i) + C(I_3/I_1)B = I_4/I_1 \quad (1)$$

$$\ddot{\theta} + (I_6/I_5)(\theta - \theta_i) + C(I_7/I_5) = I_8/I_5 \quad (2)$$

$$C = (I_3/I_9)(B^2 - B_i^2) + (I_7/I_9)(\theta^2 - \theta_i^2) - \Gamma/I_9 - I_{10}/I_9 \quad (3)$$

where, the I_i, $i = 1, \ldots, 9$ are numbers obtained from evaluation of definite integrals in the linear solution. These integrals are given in Appendix A. I_{10} is known from the prescribed boundary conditions and is taken to be zero in this paper.

Substitution of the third equation into the first two yields two equations from which B and θ may be determined. From these equations, it may be noted that no coupling exists unless large deflections are present

$$\ddot{B} + B\frac{I_2}{I_1}\left[1 - \frac{I_3}{I_2I_9}\Gamma - \frac{I_3^2}{I_2I_9}B_i^2 + \frac{I_3I_7}{I_2I_9}(\theta^2 - \theta_i^2) - \frac{I_3I_{10}}{I_2I_9}\right] + \frac{I_3^2}{I_1I_9}B^3 = \frac{I_4}{I_1} + \frac{I_2}{I_1}B_i \quad (4)$$

$$\ddot{\theta} + \theta\frac{I_6}{I_5}\left[1 - \frac{I_7}{I_6I_9}\Gamma - \frac{I_7^2}{I_6I_9}\theta_i^2 + \frac{I_3I_7}{I_6I_9}(B^2 - B_i^2) - \frac{I_7I_{10}}{I_6I_9}\right] + \frac{I_7^2}{I_5I_9}\theta^3 = \frac{I_8}{I_5} + \frac{I_6}{I_5}\theta_i \quad (5)$$

As a result of the choice of assumed functions, an examination of the linear equations will show that for $I_{10} = 0$ (Refs. 2, 3, or 4):

$$I_2/I_1 = \omega_{01}^2$$

the first mode, uniform temperature free vibration frequency;

$$I_2I_9/I_3 = \Gamma_{1\text{critical}}$$

the first mode, uncoupled, unloaded, perfect plate thermal buckling parameter for the prescribed temperature distribution;

$$I_4/I_2 = B_0$$

the first mode deflection under any applied load normal to the plate surface;

$$I_6/I_5 = \omega_{02}^2$$

the second mode, uniform temperature free vibration frequency;

$$I_6I_9/I_7 = \Gamma_{2\text{critical}}$$

the second mode, uncoupled, unloaded, perfect plate thermal buckling parameter for the prescribed temperature distribution;

$$I_8/I_6 = \theta_0$$

the second mode deflection under the applied load normal to the plate surface.

Now divide Eqs. (4) and (5) by I_2/I_1 and I_6/I_5, respectively, make the foregoing substitutions followed by the following coordinate transformations:

$$\psi^2 = (I_3/\Gamma_1)B^2,\ \phi^2 = (I_7/\Gamma_2)\theta^2$$

The resulting equations are

$$\ddot{\psi}/\omega_{01}^2 + \psi[1 - \Gamma/\Gamma_1 - \psi_i^2 + (\Gamma_2/\Gamma_1)(\phi^2 - \phi_i^2) - I_{10}/\Gamma_1] + \psi^3 = \psi_0 + \psi_i \quad (6)$$

$$\ddot{\phi}/\omega_{02}^2 + \phi[1 - \Gamma/\Gamma_2 - \phi_i^2 + (\Gamma_1/\Gamma_2)(\psi^2 - \psi_i^2) - I_{10}/\Gamma_2] + \phi^3 = \phi_0 + \phi_i \quad (7)$$

When the displacements over that part of the boundary where the displacements are prescribed are zero, the integral $I_{10} = 0$. Otherwise, it is seen that prescribing boundary displacements will cause the same trend of stiffening as prescribing Γ, ψ_i, or ϕ_i.

It is possible that I_4, I_8, and I_{10} as well as Γ may, any one or all, be functions of time, in which case the problem would be a large amplitude vibration problem under, at most, four different forcing functions.

The assumption is now made that I_4, I_8, and I_{10} are not functions of time and that the rate of change of the displacements as a result of the rate of change of Γ with time is sufficiently slow that the deflections caused by temperature may be treated as a statics problem. Thus, under heat input in the plane of the plate and load normal to the plane, the plate may undergo large static deflections about which small amplitude dynamic oscillations may occur. The square of the frequency of these oscillations is a direct measure of the effective stiffness in the mode and at the displacement under investigation.

To find the frequency of the small amplitude oscillations (see Appendix B), let

$$\psi = \epsilon_\psi + \Psi,\ \phi = \epsilon_\phi + \Phi$$

Substitution of these quantities into Eqs. (6) and (7) produces two equations for the frequencies when $(\omega_{01}/\omega_{02})^2 \ll 1$

$$(\omega/\omega_0)_1^2 = 1 - \Gamma/\Gamma_1 + 3\Psi^2 - \psi_i^2 + (\Gamma_2/\Gamma_1)(\Phi^2 - \phi_i^2) - I_{10}/\Gamma_1 \quad (8)$$

$$(\omega/\omega_0)_2^2 = 1 - \Gamma/\Gamma_2 + 3\Phi^2 - \phi_i^2 + (\Gamma_1/\Gamma_2)(\Psi^2 - \psi_i^2) - I_{10}/\Gamma_2 \quad (9)$$

Since ψ_i, ϕ_i, Γ, and I_{10} are specified quantities, it is only necessary to find Φ and Ψ in order to calculate the frequencies. Setting the inertial terms equal to zero in Eqs. (6) and (7) yields the static deflection equations

$$\Psi^3 + \Psi[1 - \Gamma/\Gamma_1 - \psi_i^2 + (\Gamma_2/\Gamma_1)(\Phi^2 - \phi_i^2) - I_{10}/\Gamma_1] = \psi_0 + \psi_i \quad (10)$$

$$\Phi^3 + \Phi[1 - \Gamma/\Gamma_2 - \phi_i^2 + (\Gamma_1/\Gamma_2)(\Psi^2 - \psi_i^2) - I_{10}/\Gamma_2] = \phi_0 + \phi_i \quad (11)$$

Therefore, the problem reduces to that of finding Φ and Ψ from the previous two equations and then substituting these results into Eqs. (8) and (9) to get the effects on frequency and stiffness. Note that the ratio of the thermal buckling eigenvalues Γ_2/Γ_1 (or $\Delta T_2/\Delta T_1$, see Appendix C) determines the degree of coupling for the large amplitude, static deflections. Both the thermal buckling eigenvalue ratio and the uniform temperature frequency ratio ω_{01}/ω_{02} influence the degree of coupling in the frequency equations. In Eqs. (8) and (9), the frequency ratio ω_{01}/ω_{02} has been assumed small, thus, its influence on the coupling has been omitted. A parametric study has been made of Eqs. (8–11) for values of thermal loading ratio Γ/Γ_2 from 0 to 2.0 and for various combinations of the parameters, $\Gamma_1/\Gamma_2 = \Delta T_1/\Delta T_2$, ϕ_i, and ψ_i, with $I_{10} = 0$.

It was found from the parametric study that for $\Delta T_1/\Delta T_2$ sufficiently greater than unity, the second mode frequency always increases after reaching a minimum as predicted in Ref. 2. However, it was also found that the first frequency levels off after reaching a minimum regardless of the magnitude of the initial deflections ψ_i and ϕ_i. But it was also found that for $\Delta T_1/\Delta T_2$ sufficiently less than unity, the first mode frequency always increases after reaching a minimum while the second mode levels off at a minimum, again independent of the magnitude of the initial deflections.

The interesting results occur when $\Delta T_1/\Delta T_2$ is sufficiently close to unity. For $\Delta T_1/\Delta T_2 = 1$, the relative magnitude of the initial displacement parameters control which mode increases and which levels off, although it is possible that

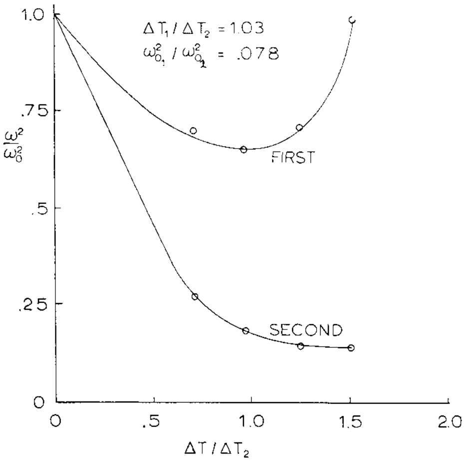

Fig. 4 Frequency response with coupling.

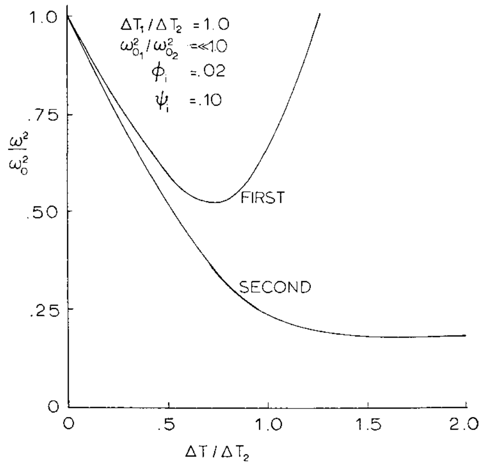

Fig. 6 Frequency response with coupling.

both may increase. For $\Delta T_1/\Delta T_2$ sufficiently close to unity, the ratio of $\Delta T_1/\Delta T_2$ combined with ψ_i and ϕ_i determines the frequency response. The effect of ω_{01}/ω_{02} remains to be investigated. A qualitative comparison of parametric study results with experimental frequency data for $\Delta T_1/\Delta T_2$ in the neighborhood of unity is shown in the following figures.

The experimental data recorded at the Air Force Institute of Technology are shown in Figs. 4, 5, 7, and 8. ψ_i and ϕ_i are not known, but the ratio $\Delta T_1/\Delta T_2$ for the rectangular plate (Fig. 4) was calculated as 1.029. Note that the first mode frequency increases while the second mode frequency levels off. Figure 5 shows the same phenomena for a slightly tapered plate.

Figure 6 shows the parametric response for the coupled first and second modes of a plate for which

$$\Delta T_1/\Delta T_2 = 1.0, \phi_i = 0.02, \psi_i = 0.1$$

$$\psi_0 = \phi_0 = I_{10} = 0, (\omega_{01}/\omega_{02})^2 \ll 1$$

The first mode frequency increases after buckling while the second mode levels off, just as in the experiment.

Figures 7 and 8 show the experimental results for highly tapered plates of the same aspect ratio and thickness as the previous plates. The calculated values of $\Delta T_1/\Delta T_2 > 1.1$ are shown in the figures. For these plates, the second mode frequency increases in the conventional manner while the first mode levels off. Again, ψ_i and ϕ_i are not known for the plates although no initial deformation was discernible by visual inspection.

Figure 9 shows that for $\Delta T_1/\Delta T_2 = 1.1$, the analytical response has the same trend as the experimental data when the initial imperfections are very small. It should be noted that the initial slope of the frequency response curves is affected by the initial deflection. Therefore, extrapolation of experimental data to obtain ΔT_1 and ΔT_2 may lead to erroneous results.

A comparison of theoretical to experimental results from Ref. 2 is shown in Fig. 3. The theoretical curve does not contain any effect of initial deformation in the bending mode, although the initial plate shape, Fig. 2, indicates quite large bending and camber. Thus, the effect of finite ψ_i is present in the experimental data.

Figure 10 shows two curves for $\Delta T_1/\Delta T_2 = 1.1$. The dashed curve is for $\phi_i = 0.06$ (about the correct value), and $\psi_i = 0$, which would be representative of the analytical curve in Fig. 3. The solid curve in Fig. 10 occurs when $\psi_i = 0.2$ and would be representative of the experimental curve in Fig. 3 which may explain in part the disagreement between experiment and theory in Ref. 2.

Conclusions

Both experimental and analytical data indicate strong coupling between the first two modes of plates with in-plane stresses. The onset of significant coupling varies with the initial imperfections and with the ratio of in-plane buckling parameters $\Delta T_1/\Delta T_2$.

For ω_{01}/ω_{02} sufficiently small, the post buckling behavior of plates for which $\Delta T_1/\Delta T_2 \neq 1$, is determined by the ratio $\Delta T_1/\Delta T_2$. If $\Delta T_1/\Delta T_2$ is sufficiently less than 1, the second mode frequency will always level off and the first mode fre-

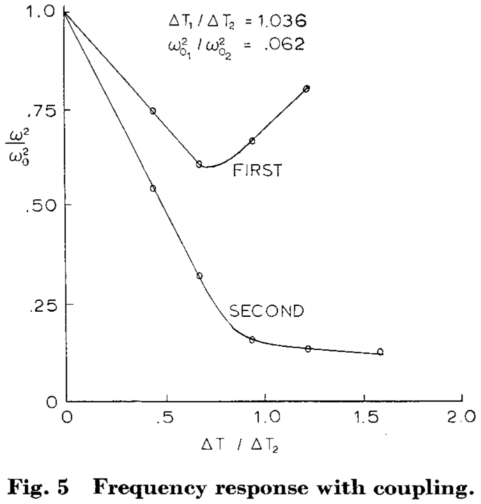

Fig. 5 Frequency response with coupling.

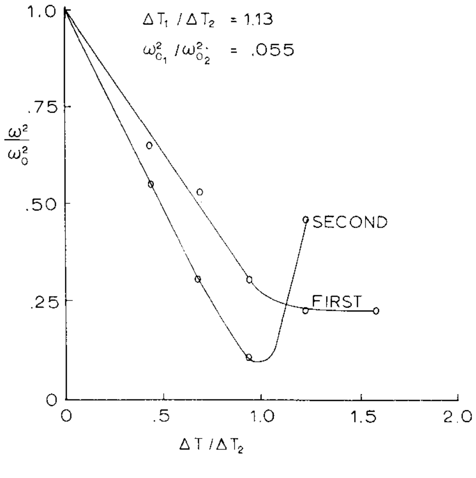

Fig. 7 Frequency response with coupling.

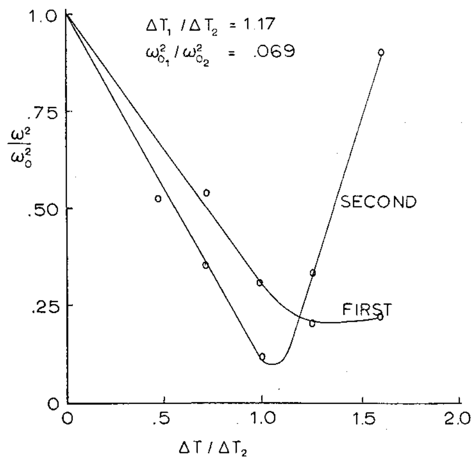

Fig. 8 Frequency response with coupling.

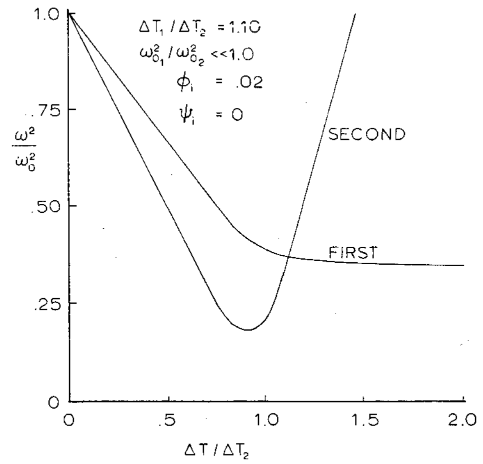

Fig. 9 Frequency response with coupling.

quency will increase, whereas for $\Delta T_1/\Delta T_2$ sufficiently greater than 1, the first mode will level off and the second mode will increase. For $\Delta T_1/\Delta T_2$ sufficiently close to 1, the relative magnitudes of the initial imperfections will determine which mode levels off and which mode increases.

Equations (8–11) and Appendix II show that the small amplitude vibration frequency about the large amplitude static equilibrium position is independent of planform shape, boundary conditions, thickness distribution and temperature distribution except as these factors affect the thermal buckling parameters and the uniform temperature frequencies.

The influence of ω_{01}/ω_{02} remains to be determined. Experimental data for quantitative comparison of theory with experiment need to be obtained including a more careful quantitative evaluation of arbitrary initial imperfection than is given in Ref. 2. Further, the essentially second-order purturbation on the linear theory presented herein should be evaluated as to its range of validity by comparison to experimental data obtained under carefully controlled conditions.

Appendix A

The I_i are nine integrals of known functions over the plate surface and are known quantities from the linear solution

$$I_1 = \iint \rho t W_1^2 dxdy$$

$$I_2 = \iint D\left\{\left(\frac{\partial^2 W_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W_1}{\partial y^2}\right)^2 + 2\nu\,\frac{\partial^2 W_1}{\partial x^2}\,\frac{\partial^2 W_1}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W_1}{\partial x\partial y}\right)^2\right\} dxdy$$

$$I_3 = \iint t\left\{\frac{\partial^2 G}{\partial x^2}\left(\frac{\partial W_1}{\partial y}\right)^2 + \frac{\partial^2 G}{\partial y^2}\left(\frac{\partial W_1}{\partial x}\right)^2 - \frac{\partial^2 G}{\partial x\partial y}\,\frac{\partial W_1}{\partial x}\,\frac{\partial W_1}{\partial y}\right\} dxdy$$

$$I_4 = \iint P W_1\, dxdy$$

$$I_5 = \iint \rho t W_2^2 dxdy$$

$$I_6 = \iint D\left\{\left(\frac{\partial^2 W_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W_2}{\partial y^2}\right) + 2\nu\,\frac{\partial^2 W_2}{\partial x^2}\,\frac{\partial^2 W_2}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W_2}{\partial x\partial y}\right)\right\}^2 dxdy$$

$$I_7 = \iint t\left\{\frac{\partial^2 G}{\partial x^2}\left(\frac{\partial W_2}{\partial y}\right)^2 + \frac{\partial^2 G}{\partial y^2}\left(\frac{\partial W_2}{\partial x}\right)^2 - 2\,\frac{\partial^2 G}{\partial x\partial y}\,\frac{\partial W_2}{\partial x}\,\frac{\partial W_2}{\partial y}\right\} dxdy$$

$$I_8 = \iint P W_2\, dxdy$$

$$I_9 = \iint \frac{t}{E}\left\{\left(\frac{\partial^2 G}{\partial x^2}\right)^2 + \left(\frac{\partial^2 G}{\partial y^2}\right)^2 - 2\nu\,\frac{\partial^2 G}{\partial x^2}\,\frac{\partial^2 G}{\partial y^2} + 2(1+\nu)\left(\frac{\partial^2 G}{\partial x\partial y}\right)^2\right\} dxdy$$

$$I_{10} = \int t\left\{\frac{\partial^2 G}{\partial x^2}\,\bar{V} + \frac{\partial^2 G}{\partial y^2}\,\bar{U}\right\} ds$$

Appendix B

The equations of motion are

$$\ddot{\psi}/\omega_{01}^2 + \psi^3 + \psi[1 - \Gamma/\Gamma_1 - \psi_i^2 + (\Gamma_2/\Gamma_1)(\phi^2 - \phi_i^2) - I_{10}/\Gamma_1] = \psi_0 + \psi_i$$

$$\ddot{\phi}/\omega_{02}^2 + \phi^3 + \phi[1 - \Gamma/\Gamma_2 - \phi_i^2 + (\Gamma_1/\Gamma_2)(\psi^2 - \psi_i^2) - I_{10}/\Gamma_0] = \phi_0 + \phi_i$$

Assume

$$\psi = \epsilon_\psi + \Psi,\ \phi = \epsilon_\phi + \Phi$$

Substitute

$$\ddot{\epsilon}_\psi/\omega_{01}^2 + (\epsilon_\psi + \Psi)^3 + [\epsilon_\psi + \Psi]\{1 - \Gamma/\Gamma_1 - \psi_i^2 + (\Gamma_2/\Gamma_1)[(\epsilon_\Phi + \Phi)^2 - \phi_i^2] - I_{10}/\Gamma_1\} = \psi_0 + \psi_i$$

$$\ddot{\epsilon}_\phi/\omega_{02}^2 + (\epsilon_\phi + \Phi)^3 + [\epsilon_\phi + \Phi]\{1 - \Gamma/\Gamma_2 - \phi_i^2 + (\Gamma_1/\Gamma_2)[(\epsilon_\psi + \Psi)^2 - \psi_i^2] - I_{10}/\Gamma_2\} = \phi_0 + \phi_i$$

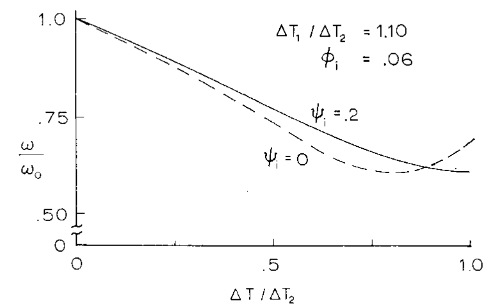

Fig. 10 Frequency response with coupling.

Neglect all second- and higher-order terms in ϵ and neglect $2\epsilon_\psi/\Psi$ and $2\epsilon_\phi/\Phi$ compared to unity. Subtract away the static deflection equation in each case to obtain

$$\ddot{\epsilon}/\omega_{0_1}{}^2 + [1 - \Gamma/\Gamma_1 + 3\Psi^2 - \psi_i{}^2 + (\Gamma_2/\Gamma_1)(\Phi^2 - \phi_i{}^2) - I_{10}/\Gamma_1]\epsilon_\psi + 2(\Gamma_2/\Gamma_1)\Psi\Phi\epsilon_\phi = 0$$

$$\ddot{\epsilon}/\omega_{0_2}{}^2 + [1 - \Gamma/\Gamma_2 + 3\Phi^2 - \phi_i{}^2 + (\Gamma_1/\Gamma_2) \times (\Psi^2 - \psi_i{}^2) - I_{10}/\Gamma_2]\epsilon_\phi + 2(\Gamma_1/\Gamma_2)\Psi\Phi\epsilon_\psi = 0$$

Assume that the small amplitude vibration will be simple harmonic (experimental evidence supports this assumption). The result of setting the determinant of the coefficients equal to zero is the frequency equation

$$(\omega^2/\omega_{0_2}{}^2)_{1,2} = \tfrac{1}{2}\{C_2 + (\omega_{0_1}{}^2/\omega_{0_2}{}^2)C_1 \mp [(C_2 - (\omega_{0_1}{}^2/\omega_{0_2}{}^2)C_1)^2 + 16(\omega_{0_1}{}^2/\omega_{0_2}{}^2)\Psi^2\Phi^2]^{1/2}\}$$

where

$$C_1 = 1 - \Gamma/\Gamma_1 + 3\Psi^2 - \psi_i{}^2 + (\Gamma_2/\Gamma_1) \times (\Phi^2 - \phi_i{}^2) - I_{10}/\Gamma_1$$

$$C_2 = 1 - \Gamma/\Gamma_2 + 3\Phi^2 - \phi_i{}^2 + (\Gamma_1/\Gamma_2)(\Psi^2 - \psi_i{}^2) - I_{10}/\Gamma_2$$

For most practical plates, the frequency of the second mode is much greater than that of the first mode. Thus, the square of the ratio $\omega_{0_1}/\omega_{0_2}$ may be assumed to be much smaller than unity causing the last term in the frequency equation to become negligibly small. Note, also, that if either Ψ or Φ are zero, this term vanishes. The frequency equation then becomes

$$(\omega^2/\omega_{0_2}{}^2)_{1,2} = \tfrac{1}{2}\{C_2 + (\omega_{0_1}{}^2/\omega_{0_2}{}^2)C_1 \mp [C_2 - (\omega_{0_1}{}^2/\omega_{0_2}{}^2)C_1]\}$$

from which

$$(\omega/\omega_0)_1{}^2 = (\omega^2/\omega_{0_2}{}^2)_1(\omega_{0_2}/\omega_{0_1})^2 = C_1 \qquad \text{(B1)}$$

$$(\omega/\omega_0)_2{}^2 = (\omega^2/\omega_{0_2}{}^2)_2{}^2 = C_2 \qquad \text{(B2)}$$

Appendix C

The thermal loading term Γ occurs as

$$\Gamma = \iint \alpha t T(x,y,\tau)(\partial_x + \partial_y)dxdy$$

Assume that the surface temperature function $T(x,y,\tau)$ may be written relative to some reference value ΔT_{ref} or

$$T(x,y,\tau) = \Delta T_{\text{ref}}(\tau)\, f(x,y)$$

Thus,

$$\Gamma = \Delta T_{\text{ref}} \iint \alpha t\, f(x,y)(\partial_x + \partial_y)dxdy$$

and

$$\Gamma_{\text{critical}} = \Delta T_{\text{ref critical}} \iint \alpha t\, f(x,y)(\partial_x + \partial_y)dxdy$$

Thus,

$$\Gamma/\Gamma_{\text{critical}} = \Delta T_{\text{ref}}/\Delta T_{\text{ref (critical)}}$$

Also

$$\Gamma_1/\Gamma_2 = \Delta T_{\text{ref (critical) first mode}}/\Delta T_{\text{ref (critical) second mode}} = \Delta T_1/\Delta T_2$$

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